

# Quantum Monte Carlo study of the transverse-field quantum Ising model on infinite-dimensional structures

Seung Ki Baek,<sup>1,2</sup> Jaegon Um,<sup>3</sup> Su Do Yi,<sup>4</sup> and Beom Jun Kim<sup>4,\*</sup>

<sup>1</sup>*Department of Physics, Sungkyunkwan University, Suwon 440-746, Korea*

<sup>2</sup>*Integrated Science Laboratory, Umeå University, S-901 87 Umeå, Sweden*

<sup>3</sup>*School of Physics, Korea Institute of Advanced Study, Seoul 130-722, Korea*

<sup>4</sup>*BK21 Physics Research Division and Department of Physics,  
Sungkyunkwan University, Suwon 440-746, Korea*

## Abstract

In a number of classical statistical-physical models, there exists a characteristic dimensionality called the upper critical dimension above which one observes the mean-field critical behavior. Instead of constructing high-dimensional lattices, however, one can also consider infinite-dimensional structures, and the question is whether this mean-field character extends to quantum-mechanical cases as well. We therefore investigate the transverse-field quantum Ising model on the globally coupled network and on the Watts-Strogatz small-world network by means of quantum Monte Carlo simulations and the finite-size scaling analysis. We confirm that both of the structures exhibit critical behavior consistent with the mean-field description. In particular, we show that the existing cumulant method has difficulty in estimating the correct dynamic critical exponent and suggest that an order parameter based on the quantum-mechanical expectation value can be a practically useful numerical observable to determine critical behavior when there is no well-defined dimensionality.

PACS numbers: 05.30.Rt, 75.10.Jm, 89.75.Hc

The effects of geometry on physical properties have drawn attention from various areas of statistical physics. First of all, we note the extensive studies on complex networks (see Ref. 1 for a review). Aside from their ubiquity and practical importance, a remarkable point is that many of the complex networks tend to enhance correlations dramatically when a statistical-physical system is put on top of them. Let us look at the Watts-Strogatz (WS) network,<sup>2</sup> for example: it starts from a one-dimensional (1D) ring of size  $N$  where each point is linked to its  $2k$  nearest neighbors. Among the total  $Nk$  bonds, we pick up  $PNk$  bonds with  $0 < P < 1$ , and then for each of them, we change one of its endpoints to a randomly chosen site. The WS network is a classical model of a small-world network characterized by two structural features: the length scale  $L$  is only logarithmic with respect to the system size  $N$  and the clustering coefficient is nevertheless relatively high.<sup>2</sup> Due to the small-world property, one observes the mean-field critical behavior in many cases, such as percolation, Ising, and  $XY$  spin models.<sup>3-6</sup> Since that is expected only for very-high-dimensional structures, many complex networks, including the WS network, are often called infinite dimensional. One may notice that this terminology also makes sense in that  $L \sim N^{1/d}$  in  $d$ -dimensional lattices while the WS network has only  $L \sim \log N$ . The theory of finite-size scaling has been well developed for such complex networks in combination with extensive numerical calculations.<sup>7</sup>

Compared to the recent progress in the study of classical systems in these structures, however, there is relatively little known about quantum cases. Even the transverse-field quantum Ising system on the Bethe lattice has been examined very recently, and it is still to be clarified whether the phase transition is really of the mean-field type,<sup>8,9</sup> and the same model on the WS network has been checked only in terms of the thermal phase transitions.<sup>10</sup> In this work, we thus investigate how the quantum Ising spin system in the transverse field behaves in the infinite-dimensional globally coupled network and the WS network. We do not consider the scale-free network in this work since the critical behavior can be possibly far richer than the conventional mean-field description.<sup>11,12</sup> The transverse-field Ising model is defined by the quantum-mechanical Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + \Gamma \sum_i \sigma_i^x, \quad (1)$$

where  $\vec{\sigma}_i \equiv (\sigma_i^x, \sigma_i^y, \sigma_i^z)$  is a spin- $\frac{1}{2}$  operator at each given site  $i$  and the summation runs over all the nearest-neighbor pairs. This Hamiltonian also contains a constant  $J > 0$  to represent the ferromagnetic-interaction strength and another constant  $\Gamma$  to represent the transverse-

field strength. If located in  $d$  dimensions, this system exhibits a phase transition at a certain critical-field strength,  $\Gamma = \Gamma_c$ , and its criticality is known to be equivalent to that of the classical  $(d + 1)$ -dimensional Ising model where  $\Gamma$  is replaced by temperature  $T$ .<sup>13–15</sup> But it is not entirely obvious what is going to happen when the dimensionality lacks its precise meaning in the infinite-dimensional structures, such as the WS network. Before studying the WS network, however, it would be good to check another infinite-dimensional structure, i.e., the globally connected network, since the transverse-field Ising system on this structure has been already known to exhibit the zero-temperature quantum phase transition of the mean-field universality by the Hamiltonian-matrix diagonalization.<sup>16</sup> The globally coupled network is simply obtained by connecting every site to all of the others, but the Hamiltonian needs a little modification to Eq. (1) in order to make it extensive as follows:

$$H = -N^{-1}J \sum_{i \neq j} \sigma_i^z \sigma_j^z + \Gamma \sum_i \sigma_i^x. \quad (2)$$

The main tool in analyzing Eqs. (1) and (2) is the world-line quantum Monte Carlo method,<sup>17</sup> where the system is endowed with an additional temporal dimension in the imaginary-time direction. The length in this direction is the same as the inverse temperature  $b \equiv 1/T$  and divided into  $N_t$  slices. The Suzuki-Trotter decomposition says that we get correct results of the original transverse-field Ising model at temperature  $T$  as  $N_t \rightarrow \infty$ . Our quantum Monte Carlo algorithm is based on the single-cluster flip algorithm<sup>18</sup> and employs the idea of continuous imaginary time, as described in Ref. 19. For convenience, we actually work with integer variables in programming (instead of floating variables in indexing) the imaginary-time axis, and then make the range of the indices,  $N_t$ , up to  $O(10^8)$ . This virtually eliminates the error involved in the width of an imaginary-time slice in the Suzuki-Trotter decomposition. We have checked that the implemented algorithm convincingly reproduces the expected results for  $d = 1$  and 2 where the dynamic critical exponent is already known to be  $z = 1$ .<sup>20</sup> The dynamic critical exponent  $z$  tells us how to scale the temporal size  $b$  with respect to  $L$  to observe correct critical behavior, so it means that  $b \sim L^z = L^1$  in ordinary  $d$ -dimensional lattices. Note that this quantity is required since our quantum Monte Carlo algorithm cannot directly access  $T = 0$ , but needs a suitable extrapolation of  $T$  in relation to the system size, unless one simulates extremely low  $T$  compared to all of the finite sizes under consideration. When  $z$  is known *a priori*, everything is just straightforward. Applying the algorithm to the infinite-dimensional structures described above, however, we usually

do not know it at the starting point and need to estimate  $z'$  to scale  $T \sim N^{-z'}$  in a suitable way. We can formulate the problem as follows: let us consider the magnetic order parameter  $|m|$  taken from the whole  $(d+1)$ -dimensional space-time region, that is

$$\langle |m| \rangle = N_t^{-1} N^{-1} \left\langle \left| \sum_{i,t} \sigma_{i,t}^z \right| \right\rangle, \quad (3)$$

where each time slice is indexed by  $t$  and the bracket  $\langle \cdots \rangle$  means the thermal average, as well as the disorder average in the case of the WS network. Note that  $\sigma_{i,t}^z$  in Eq. (3) is not an operator, but a spin value projected onto the  $z$  direction at a certain space-time. One can argue a finite-size scaling ansatz in the following two-parameter form:

$$\langle |m| \rangle = N^{-\beta/\nu'} f \left[ (\Gamma - \Gamma_c) N^{1/\nu'}, T N^{z'} \right]. \quad (4)$$

It is usually complicated to directly deal with this two-parameter scaling form. But when  $T$  is suitably scaled with a dynamic critical exponent  $z'$  so that  $\tau \equiv T N^{z'} = \text{const.}$ , Eq. (4) reduces to a usual scaling form with a single parameter, from which it is possible to determine the critical exponents. We therefore follow the procedure proposed in Refs. 21 and 22 to estimate  $z'$ . The idea is that Binder's cumulant for a finite-sized system,

$$U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}, \quad (5)$$

vanishes not only in the limit of  $T \rightarrow \infty$ , but also in the limit of  $T \rightarrow 0$  where the system virtually becomes a classical 1D Ising chain elongated in the imaginary-time direction. One may thus expect that a maximum will appear by  $U = U_{\text{max}}$  at a certain characteristic temperature  $T_{\text{max}}$  for each given system size  $N$ . If the field strength is above the critical threshold by  $\Gamma > \Gamma_c$ , then the peak value  $U_{\text{max}}$  will eventually vanish as  $N$  becomes larger, but will approach a nonzero value if  $\Gamma < \Gamma_c$  instead. This speculation provides a way to determine  $\Gamma_c$  where  $U_{\text{max}}$  remains constant regardless of  $N$ . We illustrate how it looks for the 1D case at  $\Gamma_c^{(1D)} = 1$  in Fig. 1(a). Note the nice symmetry as the size and inverse temperature  $b$  vary at the same time. It implies that one can also assume that  $U$  obeys a finite-size scaling form as

$$U = g \left[ (\Gamma - \Gamma_c) N^{1/\nu'}, T N^{z'} \right]. \quad (6)$$

Therefore,  $U$  is a function of  $\tau = T N^{z'}$  at  $\Gamma = \Gamma_c$ , and a correct  $z'$  will yield a scaling collapse of  $U$  [Fig. 1(b)]. It is also notable that  $U_{\text{max}}$  in Fig. 1 is fully consistent with

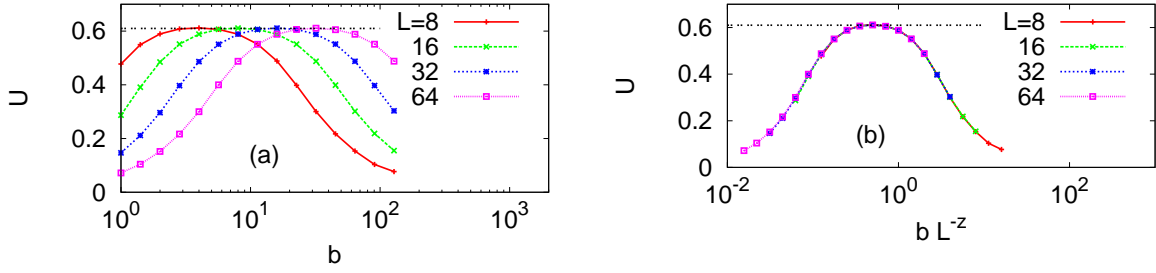


FIG. 1: (Color online) (a) Binder's cumulant for the 1D transverse-field Ising model at  $\Gamma = \Gamma_c^{(1D)} = 1$  as a function of the inverse temperature  $b = 1/T$ . The transverse-field strength  $\Gamma$  is in units of  $J$  and the temperature  $T$  is in units of  $J/k_B$  where  $k_B$  is the Boltzmann constant. (b) Scaling collapse with the dynamic critical exponent  $z = 1$ . The dotted lines indicate its universal amplitude ratio for the two-dimensional Ising model [ $U^* = 0.610692(2)$ ].

the universal amplitude ratio  $U^* = 0.610692(2)$  for the two-dimensional Ising model.<sup>23</sup> By checking the size dependence of  $b_{\max} \equiv 1/T_{\max}$  where  $U$  reaches the maximum, one will be able to obtain the dynamic critical exponent. Then, we calculate the magnetic order parameter with changing both  $N$  and  $T$ , and finally get the critical exponents in the zero-temperature thermodynamic limit by the standard finite-size scaling,  $\langle |m| \rangle \sim N^{-\beta/\nu'}$  and  $dU/d\Gamma \sim N^{1/\nu'}$  at  $\Gamma = \Gamma_c$ .

In addition, it is interesting to note that in Fig. 1(b) the curve exhibits reflection symmetry with the peak position as the axis of symmetry. The reason for this can be argued in the following way: for given  $L$ , let us begin with observing the system at  $b = b_{\max}$ . Then one can say that the correlation length in the spatial direction amounts to  $L$ , while that in the temporal direction amounts to  $b$ . Let us increase the temperature so that  $b$  becomes  $b_{\max}/2$ . When a system has different lengths in different axes, the correlation length will be bounded by the shortest one, so the correlation length will become about one half compared to the previous case of  $b = b_{\max}$ . It means that as we traverse the system in the spatial direction, we will find no correlation at a distance around  $L/2$ , and we may roughly regard the system as separated into two uncorrelated parts in the spatial direction. If we instead lower the temperature to  $b = 2b_{\max}$ , the correlation length still remains bounded by  $L$ , and we will find no correlation as we traverse the system by  $b_{\max}$  in the temporal direction. In this sense, we may regard the system as separated into two uncorrelated parts in the *temporal* direction this time. It becomes precisely symmetric to the case of  $b = b_{\max}/2$  by exchanging

the axes and adjusting the overall scale. In other words, there is symmetry between the cases of  $\log b_{\max} \rightarrow \log b_{\max} \pm \log 2$ , which explains the reflection symmetry in Fig. 1(b). If we extend this argument to a higher dimension, say  $d > 1$ , the system will be broken into  $2^d$  pieces when we take  $b \rightarrow b_{\max}/2$ . Since the other part of the argument about  $b \rightarrow 2b_{\max}$  remains the same, it predicts that  $U$  will decay faster on the low- $b$  side than on the high- $b$  side, and that this asymmetry will be more pronounced as  $d$  gets higher.

Let us check how the method using Eq. (6) works for the globally coupled network with Eq. (2). As mentioned above, it has been already established that the transverse-field quantum Ising model on it exhibits the mean-field critical behavior, and the critical threshold is at  $\Gamma_c^{(G)} = 1$  in units of  $J$ .<sup>16</sup> Since the transverse-field Ising model has the upper critical dimension  $d_u = 3$ , it also leads to  $z' = z/d_u = 1/3$ .<sup>16</sup> We thus calculate Binder's cumulant at  $\Gamma = 1$  and depict the results in Fig. 2(a). We immediately notice two features from the figure: first, the maximum height is close to the universal amplitude ratio for the mean-field model,  $U^* \approx 0.270520$ .<sup>24</sup> Second, the curve shapes become more asymmetric as  $N$  gets larger. This observation is qualitatively explained by the argument in the paragraph above, and furthermore implies that it is not possible to make a scaling collapse of this plot just by tuning  $T$  in accordance with  $N$  as postulated in Eq. (6), at least up to the sizes that we have in this work [Fig. 2(b)]. It hinders one from using the extrapolation to  $N \rightarrow \infty$  and  $T \rightarrow 0$  from the finite-sized data at finite temperatures by Eq. (6). Even in the case that one works at a fixed temperature, the finite-size effects may well enter and dominate the situation: when the temperature is relatively high, say  $T = 0.5$ , we have little difficulty finding the mean-field transition at  $\Gamma_c \approx 0.958(5)$  with the *classical* upper critical dimension  $d_u^{(\text{classic})} = 4$ , and the value of  $U$  is close to the universal amplitude ratio [Fig. 2(c) and its scaling collapse (not shown here)]. This observation can be explained as follows: rewriting Eq. (6) as  $U = g(TN^{z'})$  at  $\Gamma = \Gamma_c$ , one may expect  $g$  to converge to a constant when the system gets large enough, i.e.,  $N \gg T^{-1/z'}$ . It in turn means that the scaling form in Eq. (6) becomes insensitive to the second argument at a sufficiently high  $T$ . This is why we find such finite-size scaling behavior with varying  $\Gamma$  at  $T = 0.5$  in Fig. 2(c). The classical mean-field behavior at this  $T$  clearly confirms a consequence of crossover phenomena from the quantum to the classical critical behavior at different temperatures.<sup>25</sup> However, if we move to a low but still finite temperature, i.e.,  $T = 0.1$  in this example, the finite-size effect becomes so substantial that it gets hard to characterize the transition [Fig. 2(d)]. Note that

the crossing point is still moving to the low- $\Gamma$  side, rising up very slowly to  $U^*$ . If we increase  $N$  by more than an order of magnitude, Binder's cumulant begins to move in the correct direction [Fig. 2(e)]. The critical-field strength at  $T = 0.1$  can be estimated to differ from  $\Gamma = 1$  by less than  $5 \times 10^{-4}$ , and the cumulant will behave as  $U = U^*(1 + cN^{-\Delta} + \dots)$  with a coefficient  $c$  and a correction exponent  $\Delta \geq 0$  at this critical point. Since Eq. (6) leads to  $U = g(bN^{-z'})$  at  $\Gamma = \Gamma_c$ , one may guess that  $\Delta = z' = 1/3$  by expanding the scaling function  $g$  into a Taylor series. This simple guess is not very far from our numerical observation shown in Fig. 2(f). To sum up, our results suggest that Eq. (5) might not be a very efficient numerical observable to reproduce the correct results for infinite-dimensional structures.

As discussed above, although intuitively appealing, the concept of the characteristic temperature  $T_{\max}$  where  $U$  shows a maximum is not adequate for high-dimensional structures. The difficulty can be overcome by using an alternative quantity to  $U$ . Let us take the magnitude of spontaneous magnetization at an arbitrary *fixed* point in the imaginary-time axis,

$$\langle |s| \rangle = N^{-1} \left\langle \left| \sum_i \sigma_{i,t}^z \right| \right\rangle. \quad (7)$$

Due to the time translational symmetry,  $\langle |s| \rangle$  defined in Eq. (7) is a time-independent value, and, moreover, it corresponds to the Boltzmann statistics with eigenfunctions diagonalizing the Hamiltonian Eq. (1) or Eq. (2), i.e.,

$$\langle |s| \rangle = Z^{-1} \sum_j |s_j| e^{-bE_j}$$

where  $Z$  is the partition function, and  $|s_j| \equiv \sum_l |a_l^j|^2 |\langle s_l | \sum_i \sigma_i^z | s_l \rangle|$  and  $E_j$  are the quantum-mechanical expectation value of the operator  $\sum_i \sigma_i^z$  and the energy eigenvalue for the  $j$ th eigenstate  $|\Psi_j\rangle \equiv \sum_l a_l^j |s_l\rangle$ , respectively. Consequently, as  $T \rightarrow 0$ ,  $\langle |s| \rangle$  goes to a nonzero expectation value of the ground state for finite  $N$ , which is different from the behavior of  $\langle |m| \rangle$  in Eq. (3). The scaling form is again considered as

$$\langle |s| \rangle = N^{-\beta/\nu'} h \left[ (\Gamma - \Gamma_c) N^{1/\nu'}, TN^{z'} \right], \quad (8)$$

so one finds scaling collapse  $\langle |s| \rangle N^{\beta/\nu'} = h(TN^{z'})$  at  $\Gamma = \Gamma_c$ . Furthermore, as mentioned above, Fig. 3(a) shows that this quantity actually converges to a constant at  $b \rightarrow \infty$  for each  $N$ . Since  $\langle |s| \rangle$  is insensitive to  $T$  as  $T \rightarrow 0$ , we can drop out the second argument of the

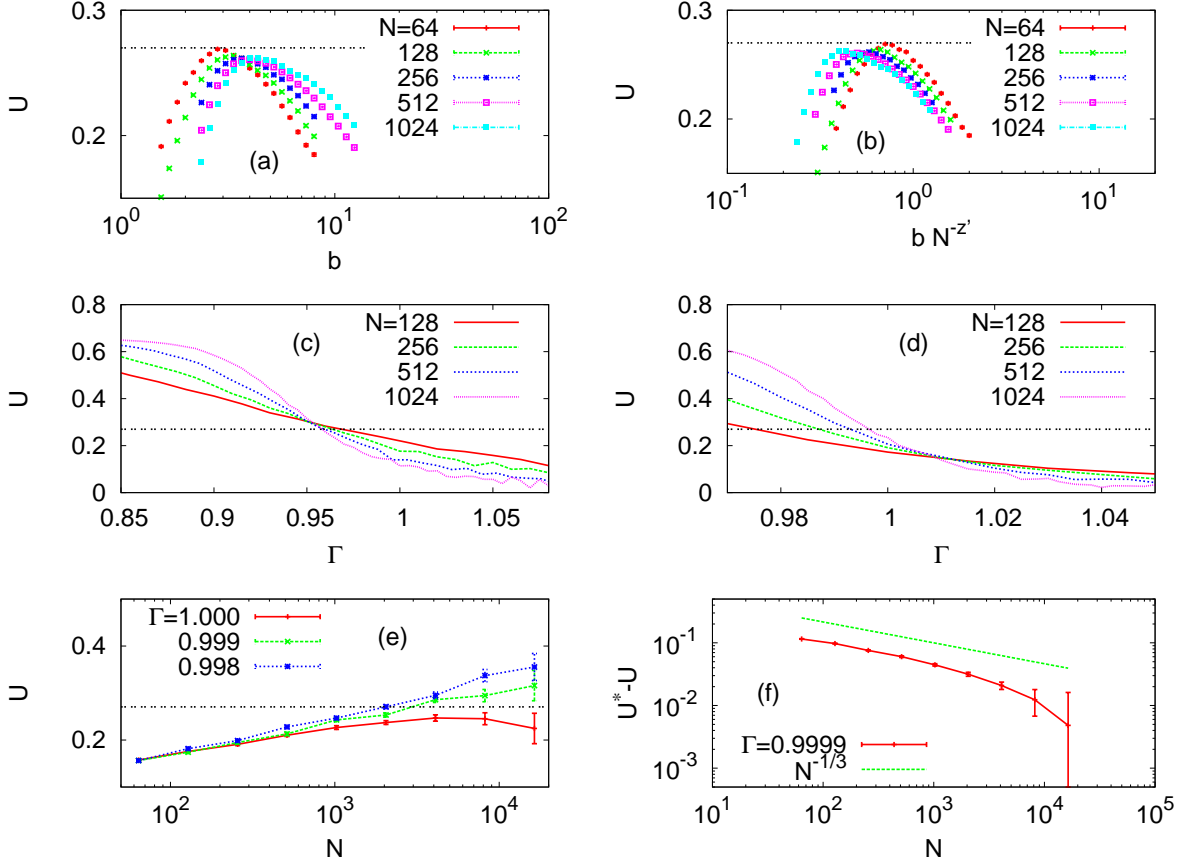


FIG. 2: (Color online) Observations for the globally coupled network. (a) Binder's cumulant at  $\Gamma = \Gamma_c^{(G)} = 1$ . (b) Scaling of the horizontal axis with the known dynamic critical exponent  $z' = 1/3$  [compare with Fig. 1(b)]. (c) Binder's cumulant at  $T = 0.5$  shows a crossing point, already close to the universal value, while (d) the crossing is still moving due to finite-size effects at  $T = 0.1$ . (e) Binder's cumulant as a function of  $N$  at three different  $\Gamma$ 's, with temperature  $T = 0.1$  as in (d). The dotted lines represent the universal amplitude ratio for the mean-field case [ $U^* \approx 0.270520$ ]. (f) The difference of  $U$  from  $U^*$  at  $\Gamma = 0.9999$ , in comparison with  $N^{-1/3}$ .

scaling function  $h$  in Eq. (8) at sufficiently low  $T$  and we are back to the single-parameter scaling form,

$$\langle |s| \rangle \approx N^{-\beta/\nu'} h \left[ (\Gamma - \Gamma_c) N^{1/\nu'} \right]$$

around  $\Gamma_c$ . Fitting the values of  $\langle |s| \rangle$  at  $T \approx 1.4 \times 10^{-2}$  as a function of  $N$ , for example, gives us an estimate of  $\beta/\nu' = 0.339(2)$ , which is fairly close to  $1/3$ . Then we can estimate  $z' \approx 0.3$  by making the data for different  $N$ 's collapse into a single curve at large  $b$  [see, e.g., Fig. 3(a)]. One might say that the mean-field behavior could have been found by using  $\langle |m| \rangle$



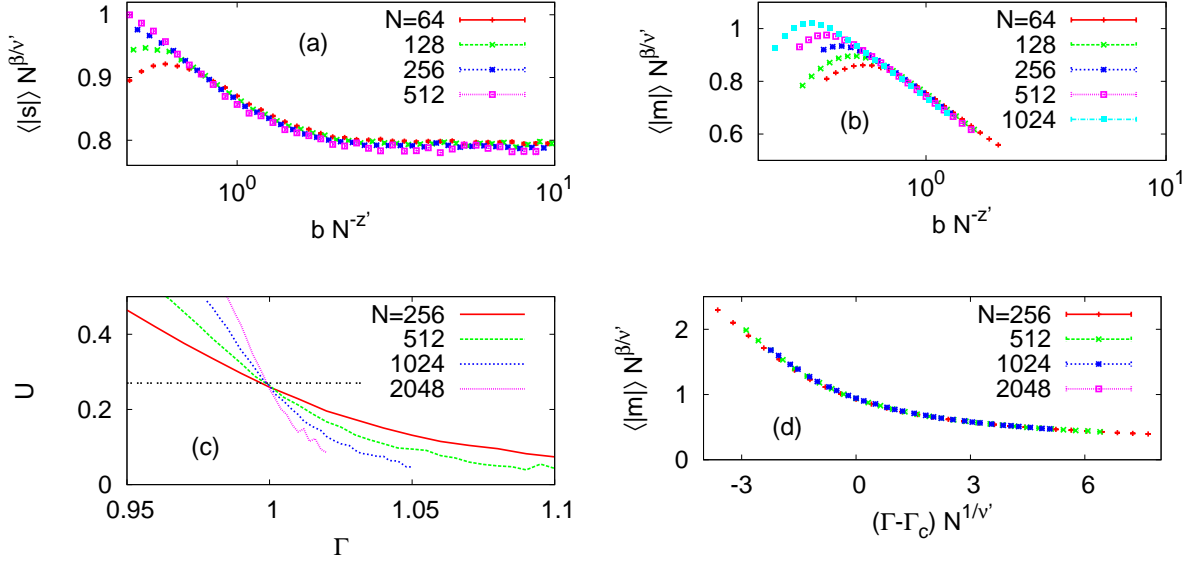


FIG. 3: (Color online) Observations for the globally coupled network. (a) Magnetization at a fixed imaginary time [Eq. (7)] yields scaling collapse with  $z' = 1/3$ . (b) Scaling of the magnetic order parameter [Eq. (3)] with  $z' = 1/3$  at  $\Gamma = 1$ . By scaling  $T$  with this  $z'$  in such a way that  $\tau = TN^{z'} = 2$ , (c) Binder's cumulant and (d) the magnetic order parameter show the correct mean-field quantum phase transition at  $\Gamma_c^{(G)} = 1$  with  $\beta/\nu' = 1/3$  and  $1/\nu' = 2/3$ . The dotted line represents the universal amplitude ratio for the mean-field case [ $U^* \approx 0.270520$ ].

instead [see Fig. 3(b)]. But our point is that  $\langle |s| \rangle$  is easier to handle in the sense that it converges a well-defined value in the large- $b$  limit, which allows us to get around the problem of working with two scaling parameters at the same time. Beginning with scaling  $\langle |s| \rangle$ , one finds numerical observables such as  $U$  and  $\langle |m| \rangle$  consistent with the mean-field behavior [Figs. 3(c) and 3(d)]. Therefore, we suggest that Eq. (8) can be a practically useful guidance in measuring the dynamic critical exponent. The advantage of using  $U$ , on the other hand, comes from its universal values containing a good deal of information on universality classes, and it indeed works well in estimating  $\Gamma_c$ . Figures 3(a) and 3(b) furthermore suggest that a relevant scaling region should be found at  $b \gg 1$ , rather than around the peak position. Once again, we note that  $U$  vanishes at this large  $b$ , while  $\langle |s| \rangle$  does not vanish for finite  $N$ , so that the latter quantity can be more easily handled at the scaling region.

Let us now proceed to the WS network. We choose  $k = 3$  and  $P = 0.05$  and take disorder averages over independent realizations. The number of realizations varies but usually

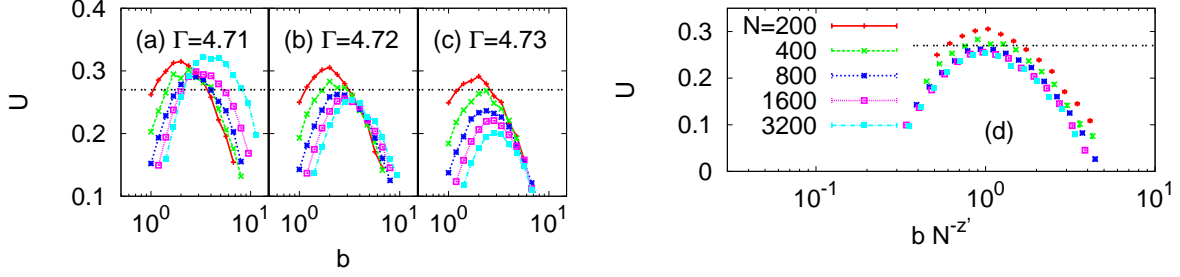


FIG. 4: (Color online) Observations for the WS network. (a)–(c) Binder’s cumulant as a function of  $b$  at three values of  $\Gamma$ . In each panel, the curves represent  $N = 200, 400, 800, 1600$ , and  $3200$  from left to right, respectively. (d) From Eq. (6), we measure the dynamic critical exponent  $z' = 0.197(8)$ , whereby this scaling collapse is obtained. The dotted lines represent the universal amplitude ratio for the mean-field case [ $U^* \approx 0.270520$ ].

amounts to  $O(10^3)$ . We begin with the existing method using  $U$ , assuming that it will work here as in 1D. Then we find characteristic temperatures for various sizes of the WS network by measuring  $U$ , from which we estimate  $\Gamma_c^{(\text{WS})} = 4.72(1)$ . [Figs. 4(a) to 4(c)]. Noting that the small sizes tend to overestimate  $U$ , we determine  $\Gamma_c$  by observing the largest sizes. It is interesting that we do not see such large asymmetry as in the globally coupled network [compare Fig. 2(a) with Fig. 4(d)], which might be due to the fact that the length scale of the WS network is a logarithmic function of  $N$ . Another interesting indication is that the height of  $U_{\text{max}}$  at  $\Gamma = 4.72$  is close to the mean-field universal amplitude ratio [Figs. 4(b) and 4(d)]. The dynamic critical exponent at  $\Gamma = 4.72$  is estimated as  $z' = 0.197(8)$  from the dependence of  $T_{\text{max}}$  on  $N$  [Fig. 4(d)]. It is apparently very close to  $1/5$ , and leads to  $\beta/\nu' = 0.296(8)$  and  $1/\nu' = 0.598(16)$ , or, in other words,  $\beta = 0.49(3)$  and  $\nu' = 1.67(4)$ . Although they are close to the mean-field values  $1/2$  and  $3/2$ , respectively, we have no explanation for  $z' = 1/5$ .

Since we already know that Eq. (6) does not always give correct  $z'$ , we may instead try the alternative method relying upon Eq. (8). By fitting the values of  $\langle |s| \rangle$  at  $T \approx 1.9 \times 10^{-2}$  with different  $N$ ’s, we find  $\beta/\nu' = 0.33(1)$ , which contains  $1/3$  within the error bar. The dynamic critical exponent is then estimated as  $z' \approx 0.3$  by making the data collapse [see, e.g., Fig. 5(a)]. This method definitely prefers  $z' = 1/3$  to  $1/5$ , which implies that the use of Eq. (6) might yield a spurious estimate of  $z'$  as in the globally coupled network. As seen in Fig. 5(a), our observations of  $\beta/\nu'$  and  $z'$  are fully consistent since both of them are

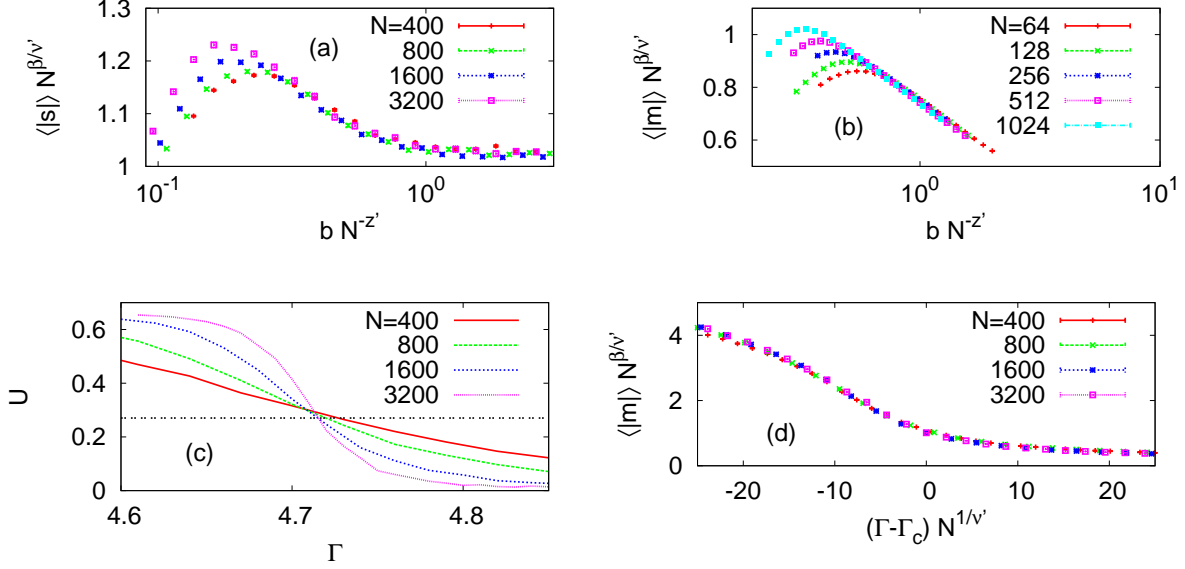


FIG. 5: (Color online) Observations for the WS network. (a) Scaling collapse of  $\langle |s| \rangle$  at  $\Gamma = \Gamma_c^{(\text{WS})} = 4.72$  according to Eq. (8), with  $\beta/\nu' = 1/3$  and  $z' = 1/3$ . (b) Scaling collapse of the magnetic order parameter with  $z' = 1/3$ . Choosing this  $z'$ , we find (c) the crossing of  $U$  and (d) scaling collapse of  $\langle |m| \rangle$  at  $\Gamma = 4.72$  with the mean-field values, i.e.,  $\beta/\nu' = 1/3$  and  $1/\nu' = 2/3$ . The dotted line indicates the universal amplitude ratio for the mean-field case [ $U^* \approx 0.270520$ ].

explained by  $d_u = 3$ . Note also the nice agreement in Fig. 5(b). Accepting this  $z' = 1/3$ , the crossing point of  $U$  is also consistent with the universal amplitude ratio of the mean-field type, and the best fits with  $\langle |m| \rangle$  and  $dU/d\Gamma$  give us  $\beta = 0.50(3)$  and  $\nu' = 1.38(4)$  [Fig. 5(c)]. The value of  $\nu'$  is reasonably close to  $3/2$  and our numerical results do not rule out this value [Fig. 5(d)]. Overall, we may consider our Monte Carlo results for the WS network as supporting the mean-field behavior, i.e.,  $\beta = 1/2$ ,  $\nu' = 3/2$ , and  $U^* \approx 0.270520$ . It is worth noting that Eq. (8) again outperforms Eq. (6) in the finite-size scaling analysis.

In summary, we have studied the transverse-field Ising model on infinite-dimensional structures by quantum Monte Carlo simulations. Since the world-line Monte Carlo algorithm cannot directly access  $T = 0$ , we need the dynamic critical exponent  $z'$  for each of the structures. For the WS network, we have estimated  $z' = 0.3(1)$  and  $\beta/\nu' = 0.33(1)$  from  $\langle |s| \rangle$ , or  $\beta = 0.50(3)$  and  $\nu' = 1.38(4)$  from  $\langle |m| \rangle$  and  $dU/d\Gamma$  under the assumption that  $z' = 1/3$ . These strongly support the mean-field behavior with the upper critical dimension  $d_u = 3$ , i.e.,  $\beta = 1/2$  and  $\nu' = 3/2$ . Drawing this conclusion, we have also found that the

existing finite-size scaling method using  $U$  can give incorrect values of  $z'$  in the infinite-dimensional structures. For this reason, we have suggested an alternative observable  $\langle |s| \rangle$  based on the quantum-mechanical expectation value, which is found to converge more quickly to the correct scaling behavior than  $U$  does.

## Acknowledgments

We thank Hangmo Yi for discussions and comments. This work was supported by a National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (Grant No. 2011-0015731). This research was conducted using the resources of the High Performance Computing Center North (HPC2N). We thank R. Orús and J. Vidal for informing us of relevant works on the Lipkin-Meshkov-Glick model.<sup>26</sup>

---

\* Corresponding author, E-mail: beomjun@skku.edu

- <sup>1</sup> S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Rev. Mod. Phys.* **80**, 1275 (2008).
- <sup>2</sup> D. J. Watts and S. H. Strogatz, *Nature (London)* **393**, 440 (1998).
- <sup>3</sup> C. Moore and M. E. J. Newman, *Phys. Rev. E* **62**, 7059 (2000).
- <sup>4</sup> B. J. Kim, H. Hong, P. Holme, G. S. Jeon, P. Minnhagen, and M. Y. Choi, *Phys. Rev. E* **64**, 056135 (2001).
- <sup>5</sup> H. Hong, B. J. Kim, and M. Y. Choi, *Phys. Rev. E* **66**, 018101 (2002).
- <sup>6</sup> K. Medvedyeva, P. Holme, P. Minnhagen, and B. J. Kim, *Phys. Rev. E* **67**, 036118 (2003).
- <sup>7</sup> H. Hong, M. Ha, and H. Park, *Phys. Rev. Lett.* **98**, 258701 (2007).
- <sup>8</sup> D. Nagaj, E. Farhi, J. Goldstone, P. Shor, and I. Sylvester, *Phys. Rev. B* **77**, 214431 (2008).
- <sup>9</sup> F. Krzakala, A. Rosso, G. Semerjian, and F. Zamponi, *Phys. Rev. B* **78**, 134428 (2008).
- <sup>10</sup> H. Yi and M.-S. Choi, *Phys. Rev. E* **67**, 056125 (2003).
- <sup>11</sup> H. Yi, *Eur. Phys. J. B* **61**, 89 (2008).
- <sup>12</sup> H. Yi, *Phys. Rev. E* **81**, 012103 (2010).
- <sup>13</sup> R. J. Elliott, P. Pfeuty, and C. Wood, *Phys. Rev. Lett.* **25**, 443 (1970).
- <sup>14</sup> P. Pfeuty, *J. Phys. C* **9**, 3993 (1976).
- <sup>15</sup> J. Um, S.-I. Lee, and B. J. Kim, *J. Korean Phys. Soc.* **50**, 285 (2007).

- <sup>16</sup> R. Botet and R. Jullien, Phys. Rev. B **28**, 3955 (1983).
- <sup>17</sup> N. Kawashima and K. Harada, J. Phys. Soc. Jpn **73**, 1379 (2004).
- <sup>18</sup> U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).
- <sup>19</sup> H. Rieger and N. Kawashima, Eur. Phys. J. B **9**, 233 (1999).
- <sup>20</sup> S. Sachdev, *Quantum phase transitions* (Cambridge University Press, Cambridge, 1999).
- <sup>21</sup> M. Guo, R. N. Bhatt, and D. A. Huse, Phys. Rev. Lett. **72**, 4137 (1994).
- <sup>22</sup> H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994).
- <sup>23</sup> J. Salas and A. D. Sokal, J. Stat. Phys. **98**, 551 (2000).
- <sup>24</sup> E. Luijten and H. W. J. Blöte, Int. J. Mod. Phys. C **6**, 359 (1995).
- <sup>25</sup> W. A. C. Erkelens, L. P. Regnault, J. Rossat-Mignod, J. E. Moore, R. A. Butera, and L. J. de Jongh, EPL **1**, 37 (1986).
- <sup>26</sup> S. Dusuel and J. Vidal, Phys. Rev. Lett. **93**, 237204 (2004); P. Ribeiro, J. Vidal, and R. Mosseri, *ibid.* **99**, 050402 (2007); R. Orús, S. Dusuel, and J. Vidal, *ibid.* **101**, 025701 (2008).